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THE LAW OF THE ITERATED LOGARITHM

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To Leta

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CHAPTER I

INTRODUCTION

The mathematical theory of probability is concerned with laws of chance. For example, the "laws of large numbers" are concerned with showing that the average value of a large number of independent (or weakly dependent) random variables is, in a sense to be made precise, almost constant. The "limit distribution theorems" show that the probability distribution of various functions of random variables, such as suitably normed sums, tends to a limiting probability distribution as the number of random variables increases. Another type of law of chance is concerned with the delicate problem of analyzing regularities of chance fluctuations. This thesis is concerned with this latter type of law of chance. In particular, it is concerned with the law of the iterated logarithm. To motivate discussion of the law, a simple example will be useful.

Consider a sequence $\{X_n\}$ of independent random variables, each with range $\{0, 1\}$, and such that

$$P[X_n = 0] = P[X_n = 1] = \frac{1}{2}$$

for each positive integer n. If we consider the successive terms X_n as representing the results of a sequence of independent tosses of a fair coin, then $X_n = 1$ corresponds to heads on the nth toss, and $X_n = 0$ corresponds to tails on the nth toss. Consequently the random variable

$$S_n = X_1 + X_2 + \ldots + X_n$$

measures the total number of heads in n tosses. In this example, the strong law of large numbers asserts that $\frac{S_n}{n}$ has limit one half almost surely, as $n \to \infty$. Equivalently, $S_n \sim \frac{n}{2}$ with probability one as $n \to \infty$. Here, $\sim$ indicates asymptotic equality: the ratio of the two sides tends to one almost surely, as $n \to \infty$. For convenience, let $Y_n = 2X_n - 1$ for each n. Then

$$T_n = Y_1 + Y_2 + \ldots + Y_n = 2S_n - n = 2(S_n - \frac{n}{2})$$

denotes the <u>net gain</u> after n tosses of the player who bets on heads before each coin toss, since $Y_n = -1$ when $X_n = 0$, and $Y_n = 1$ when $X_n = 1$. For the sequence $\{Y_n\}$, the strong law of large numbers asserts that $\frac{T_n}{n} \to 0$ almost surely as $n \to \infty$. The law of the iterated logarithm in this special case is concerned with exceptionally large values of T_n, and in this special case was proved by Khinchin in 1924. It asserts specifically that

$$\limsup_{n \to \infty} \frac{T_n}{\sqrt{2 \log \log n}} = 1$$

with probability one. This means that if ε is given, with $0 < \varepsilon < 1$, the probability is one that only finitely many of the events

$$T_n > \sqrt{2n \log \log n}\,(1 + \varepsilon)$$

occur, while the probability is one that infinitely many of the events

$$T_n < \sqrt{2 \log \log n}\,(1 - \varepsilon)$$

occur. Khinchin's proof, to be found in Khinchin [5], requires quite delicate reasoning. Related problems were treated by Hausdorff [4] and Hardy and Littlewood [3], and the Khinchin theorem improved on the earlier results. In 1929, a more general law of the iterated logarithm was proved by Kolmogorov [6], and this in turn was later generalized by Feller [2], Strassen [8], and others.

In this thesis, the principal aim is to give a reasonably self-contained proof of Kolmogorov's version of the theorem, following the lines of Kolmogorov's proof, but with many added details. Following this, some of the later generalizations will be discussed, usually without proofs. The work of Strassen [8] in 1964 throws considerable light on the meaning of the theorem, in relation to Brownian paths, but the technical background is very extensive. The bibliography contains a fairly complete list of references concerned with the law of the iterated logarithm.

In order to keep the thesis reasonable in length, some relevant parts of the general theory of probability have been assumed. Appropriate references are given, e.g., to the Borel-Cantelli lemmas, and similar results.

CHAPTER II

PROOF OF THE LAW OF THE ITERATED LOGARITHM

This chapter contains the proof of the law of the iterated logarithm (Theorem 3) together with several preliminary lemmas and theorems. Only Theorems 1 and 2 and Lemma 6 are explicitly referred to in the proof of Theorem 3. The remaining lemmas are used only to prove the lemmas and theorems that follow them.

<u>Lemma 1</u>

If $z \geq 0$, then $\exp[z(1 - z)] \leq 1 + z$.

Proof. Consider the function

$$g(z) = \log(1 + z) - z + z^2, \text{ for } z \geq 0.$$

Then

$$g'(z) = \frac{1}{1 + z} - 1 + 2z = \frac{z + 2z^2}{1 + z} > 0 \text{ if } z > 0.$$

Since $g(0) = 0$, it follows that

$$\log(1 + z) \geq z(1 - z) \text{ if } z \geq 0.$$

Hence

$$\exp[z(1 - z)] \leq 1 + z \quad \text{if } z \geq 0.$$

Lemma 2

Let X be a random variable with E(X) = 0, and let c be a positive real number such that $P[|X| \leq c] = 1$. Let σ^2 denote the variance of X. Then, if $0 < t \leq \frac{1}{c}$, we have

$$E[\exp(tX)] < \exp \frac{t^2\sigma^2}{2}(1 + \frac{tc}{2}) \tag{1}$$

and

$$E[\exp(tX)] > \exp \frac{t^2\sigma^2}{2}(1 - tc). \tag{2}$$

Proof. To prove (1), note that, since exp(tX) is dominated by exp(tc) on (-c,c),

$$E[\exp(tX)] = E(1) + E(tX) + E(\frac{t^2X^2}{2!}) + E(\frac{t^3X^3}{3!}) + \ldots$$

$$\leq 1 + \frac{t^2\sigma^2}{2}(1 + \frac{tc}{3} + \frac{t^2c^2}{3\cdot 4} + \ldots)$$

$$= 1 + \frac{t^2\sigma^2}{2}[1 + \frac{tc}{3}(1 + \frac{tc}{4} + \ldots)].$$

Since $0 < tc \leq 1$ and $\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}$, we have that

$$E[\exp(tX)] < 1 + \frac{t^2\sigma^2}{2}\left[1 + \frac{tc}{3}\left(\sum_{n=0}^{\infty} \frac{1}{4^n}\right)\right]$$

$$= 1 + \frac{t^2\sigma^2}{2}\left(1 + \frac{4tc}{9}\right)$$

$$< 1 + \frac{t^2\sigma^2}{2}\left(1 + \frac{tc}{2}\right).$$

Since $1 + z \leq e^z$ for $z \geq 0$, it follows that

$$E[\exp(tX)] < \exp \frac{t^2\sigma^2}{2}\left(1 + \frac{tc}{2}\right) .$$

This completes the proof of (1). To prove (2), note that

$$E(X^n) \geq -c^{n-2}E(X^2) \quad \text{if } n \geq 3.$$

Then, for $0 < tc \leq 1$, we have

$$E[\exp(tX)] \geq 1 + \frac{t^2\sigma^2}{2}\left(1 - \frac{tc}{3} - \frac{t^2c^2}{3\cdot 4} - \ldots\right).$$

By an argument similar to that used to prove (1), it follows that

$$E[\exp(tX)] > 1 + \frac{t^2\sigma^2}{2}\left(1 - \frac{tc}{2}\right).$$

By Lemma 1,

$$1 + z \geq \exp[z(1 - z)] \quad \text{for } z \geq 0.$$

Consequently

$$E[\exp(tX)] > \exp\{\frac{t^2\sigma^2}{2}(1 - \frac{tc}{2})[1 - \frac{t^2\sigma^2}{2}(1 - \frac{tc}{2})]\}$$

$$> \exp[\frac{t^2\sigma^2}{2}(1 - \frac{tc}{2} - \frac{t^2\sigma^2}{2})]$$

$$\geq \exp[\frac{t^2\sigma^2}{2}(1 - tc)],$$

since $0 \leq \sigma^2 \leq c^2$ and $0 < tc \leq 1$. This proves (2).

<u>Lemma 3</u>

Let $X_1, \ldots, X_n$ denote n independent random variables such that $E(X_k) = 0$, $1 \leq k \leq n$ and $\mathrm{Var}(X_k) > 0$ for all $k \geq 1$. Let $d_1, \ldots, d_n$ be n positive real numbers such that

$$P[|X_k| \leq d_k] = 1$$

for $1 \leq k \leq n$. Let $S_n = X_1 + \ldots + X_n$, and let s_n^2 denote the variance of S_n. Also, let $c_n = \max_{1 \leq k \leq n} \{\text{ess sup} \frac{|X_k|}{s_n}\}$, and suppose that $c_n > 0$ for all $n \geq 1$. Then, if $0 < t \leq \frac{1}{c_n}$,

$$\exp[\frac{t^2}{2}(1 - tc_n)] < E(\exp[\frac{tS_n}{s_n}]) < \exp[\frac{t^2}{2}(1 + \frac{tc_n}{2})].$$

Proof. Let σ_k^2 denote the variance of X_k, $1 \leq k \leq n$. Since $X_1, \ldots, X_n$ are independent,

$$E(\exp[\frac{tS_n}{s_n}]) = \prod_{k=1}^{n} E(\exp[\frac{tX_k}{s_n}]).$$

From Lemma 2 (applied to $\frac{X_k}{s_n}$), we have

$$\exp[\frac{\sigma_k^2}{2s_n^2} t^2(1 - tc_n)] < E(\exp[\frac{tX_k}{s_n}]) < \exp[\frac{t^2\sigma_k^2}{2s_n^2}(1 + \frac{tc_n}{2})]$$

for $1 \leq k \leq n$. Thus

$$\prod_{k=1}^{n} \exp[\frac{\sigma_k^2}{2s_n^2} t^2(1 - tc_n)] < \prod_{k=1}^{n} E(\exp[\frac{tX_k}{s_n}])$$

$$< \prod_{k=1}^{n} \exp[\frac{t^2\sigma_k^2}{2s_n^2}(1 + \frac{tc_n}{2})],$$

or,

$$\exp[\frac{t^2}{2s_n^2}(1 - tc_n)(\sum_{k=1}^{n} \sigma_k^2)] < E(\exp[\frac{tS_n}{s_n}])$$

$$< \exp\left[\frac{t^2}{2s_n^2}\left(1 + \frac{tc_n}{2}\right)\left(\sum_{k=1}^{n} \sigma_k^2\right)\right].$$

Since $\sum_{k=1}^{n} \sigma_k^2 = s_n^2$, it follows that

$$\exp\left[\frac{t^2}{2}(1 - tc_n)\right] < E\left(\exp\left[\frac{tS_n}{s_n}\right]\right) < \exp\left[\frac{t^2}{2}\left(1 + \frac{tc_n}{2}\right)\right].$$

<u>Theorem 1</u>

Let $X_1, \ldots, X_n$ denote n independent random variables such that $E(X_k) = 0$, $1 \leq k \leq n$ and $\mathrm{Var}(X_k) > 0$ for all $k \geq 1$. Let $d_1, \ldots, d_n$ be n positive real numbers such that

$$P[|X_k| \leq d_k] = 1$$

for $1 \leq k \leq n$. Let $S_n = X_1 + \ldots + X_n$, and let s_n^2 denote the variance of S_n. Also, let $c_n = \max_{1 \leq k \leq n} \{\text{ess sup} \frac{|X_k|}{s_n}\}$ and suppose $c_n > 0$ for all $n \geq 1$. Then

$$\text{If } 0 < \varepsilon < \frac{1}{c_n}, \text{ then } P\left[\frac{S_n}{s_n} > \varepsilon\right] < \exp\left[-\frac{\varepsilon^2}{2}\left(1 - \frac{\varepsilon c_n}{2}\right)\right] \tag{3}$$

and

$$\text{If } \varepsilon \geq \frac{1}{c_n} > 0, \text{ then } P\left[\frac{S_n}{s_n} > \varepsilon\right] < \exp\left(-\frac{\varepsilon}{4c_n}\right). \tag{4}$$

Proof. If I_A is the indicator of event A, then

$$E\{\exp \frac{tS_n}{s_n}\} \geq E\{I_{[S_n/s_n > \varepsilon]} \exp \frac{tS_n}{s_n}\} \geq \exp(t\varepsilon) P[\frac{S_n}{s_n} > \varepsilon].$$

Combining this result with the second inequality in Lemma 3 gives, if $0 < t \leq \frac{1}{c_n}$,

$$P[\frac{S_n}{s_n} > \varepsilon] \leq \exp(-t\varepsilon) E\{\exp \frac{tS_n}{s_n}\} < \exp\{-t\varepsilon + \frac{t^2}{2}(1 + \frac{tc_n}{2})\}.$$

If $0 < \varepsilon < \frac{1}{c_n}$, let $t = \varepsilon$. Thus

$$P[\frac{S_n}{s_n} > \varepsilon] < \exp\{-\varepsilon^2 + \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{4} c_n\} = \exp\{-\frac{\varepsilon^2}{2}(1 - \frac{\varepsilon c_n}{2})\},$$

which proves (3). If $\varepsilon \geq \frac{1}{c_n}$, let $t = \frac{1}{c_n}$, so that $0 < tc_n \leq 1$. Then

$$P[\frac{S_n}{s_n} > \varepsilon] < \exp\{-\frac{\varepsilon}{c_n} + \frac{1}{2c_n^2} + \frac{1}{4c_n^2}\}$$

$$\leq \exp\{-\frac{\varepsilon}{c_n} + \frac{3\varepsilon}{4c_n}\}$$

$$= \exp\{-\frac{\varepsilon}{4c_n}\},$$

which proves (4).

Theorem 2

Let $X_1, \ldots, X_n$ be n independent random variables such that $E(X_k) = 0$, $1 \leq k \leq n$ and $\mathrm{Var}(X_k) > 0$ for all $k \geq 1$. Let $d_1, \ldots, d_n$ be n positive real numbers such that

$$P[|X_k| \leq d_k] = 1$$

for $1 \leq k \leq n$. Let $S_n = X_1 + \ldots + X_n$, and let s_n^2 denote the variance of S_n. Let $c_n = \max_{1 \leq k \leq n} \{\text{ess sup } |X_k|/s_n\}$. If

$$\frac{xc_n}{s_n} = \omega < \frac{1}{64^2}, \tag{5}$$

$$\frac{x^2}{s_n^2} = \lambda > 2^{14}, \tag{6}$$

and

$$\varepsilon = \max\{32\sqrt{\frac{\log \lambda}{\lambda}}, 64\sqrt{\omega}\},$$

then

$$P[S_n > x] > \exp[-\frac{x^2}{2s_n^2}(1 + \varepsilon)].$$

Proof. Let $\delta = \frac{\varepsilon}{8}$. Then

$$\delta^2 = \max\{16 \frac{\log \lambda}{\lambda}, 64\omega\}. \tag{7}$$

Since $\frac{\log \lambda}{\lambda}$ decreases as λ increases, we have from (6)

$$16 \frac{\log \lambda}{\lambda} < 16 \frac{14\log 2}{2^{14}} < \frac{2^5 \cdot 7(0.7)}{2^{14}} = \frac{4.9}{2^9} < \frac{1}{2^6} = \frac{1}{64}.$$

From (5) we have $64\omega < \frac{1}{64}$. In either case, from (7) we have $\delta^2 < \frac{1}{64}$, or equivalently

$$\delta < \frac{1}{8}. \tag{8}$$

Let $a = \frac{x}{s_n^2(1 - \delta)}$. Then

$$x = as_n^2(1 - \delta), \tag{9}$$

$$\frac{x}{s_n^2} < a < \frac{2x}{s_n^2}, \tag{10}$$

$$ac_n s_n < 2\omega < \frac{1}{128}, \tag{11}$$

$$a^2 s_n^2 > \lambda > 512. \tag{12}$$

From (11) we have $0 < ac_n s_n < 1$. Hence, from the first inequality in Lemma 3 (letting $t = s_n a$), it follows that

$$E[\exp(aS_n)] > \exp[\frac{a^2 s_n^2}{2}(1 - ac_n s_n)].$$

Since $ac_n s_n < \frac{\delta^2}{4}$ from (11) and (7), we have

$$E[\exp(aS_n)] > \exp[\frac{a^2 s_n^2}{2}(1 - \frac{\delta^2}{4})] . \tag{13}$$

Now let $q(y) = P[S_n > y]$. Then integrating by parts shows that

$$E[\exp(aS_n)] = \int_{-\infty}^{+\infty} \exp(ay)\,dq(y)$$

$$= -\exp(ay)q(y)\Big|_{-\infty}^{+\infty} + a\int_{-\infty}^{+\infty} \exp(ay)q(y)\,dy.$$

Since S_n is bounded, $q(y) = 0$ for y sufficiently large. Hence

$$E[\exp(aS_n)] = a\int_{-\infty}^{+\infty} \exp(ay)q(y)\,dy \tag{14}$$

$$= a\left(\int_{-\infty}^{0} + \int_{0}^{as_n^2(1-\delta)} + \int_{as_n^2(1-\delta)}^{as_n^2(1+\delta)} + \int_{as_n^2(1+\delta)}^{8as_n^2} + \int_{8as_n^2}^{+\infty}\right)$$

$$= a\left(\int_{I_1} + \int_{I_2} + \int_{I_3} + \int_{I_4} + \int_{I_5}\right)$$

$$= a\,(\,J_1 + J_2 + J_3 + J_4 + J_5\,).$$

Since $q(y) \leq 1$, it follows that

$$aJ_1 \leq a\int_{-\infty}^{0} \exp(ay)\,dy = 1. \tag{15}$$

We now estimate aJ_5. Note first that from (5) and (10) it follows that

$$\frac{s_n}{xc_n} > 64^2 > 16 > \frac{8as_n^2}{x},$$

and hence that

$$\frac{s_n}{c_n} > 8as_n^2 .$$

If $y \geq \frac{s_n}{c_n}$, then setting $y = \varepsilon s_n$ in Theorem 1 (4) yields, since $\varepsilon \geq \frac{1}{c_n}$,

$$q(y) < \exp(-\frac{y}{4c_n s_n}).$$

From (11) it follows that

$$q(y) < \exp(-2ay) \quad \text{for } y \geq \frac{s_n}{c_n} .$$

On the other hand, if $8as_n^2 \le y < \frac{s_n}{c_n}$, then $0 < \frac{yc_n}{s_n} < 1$, and setting $y = \varepsilon s_n$ in Theorem 1 (3) yields

$$q(y) < \exp[-\frac{y^2}{2s_n^2}(1 - \frac{yc_n}{2s_n})].$$

Since $0 < \frac{yc_n}{2s_n} < \frac{1}{2}$, we have $1 - \frac{yc_n}{2s_n} > \frac{1}{2}$. Hence

$$q(y) < \exp(-\frac{y^2}{4s_n^2}).$$

Since $8as_n^2 \le y < \frac{s_n}{c_n}$, it follows that $\frac{-y}{4s_n^2} \le -2a$. Thus $q(y) < \exp(-2ay)$ for $8as_n^2 \le y < \frac{s_n}{c_n}$. Hence for any y in I_5 we have $q(y) < \exp(-2ay)$. It now follows that

$$aJ_5 < (a)\int_{8as_n^2}^{+\infty} \exp(-ay)\,dy < 1. \tag{16}$$

From (13) we have

$$E[\exp(aS_n)] > \exp \frac{a^2 s_n^2}{2}(1 - \frac{\delta^2}{4}) > \exp[256(1 - \frac{\delta^2}{4})]$$

$$> \exp[256(\frac{255}{256})] = \exp(255) > 8 ,$$

where the second inequality follows from (12) and the third inequality follows from (8). Thus

$$2 < \frac{1}{4} E[\exp(aS_n)].$$

From (15) and (16) we have $aJ_1 + aJ_5 < 2$. Hence

$$aJ_1 + aJ_5 < \frac{1}{4} E[\exp(aS_n)]. \tag{17}$$

Now consider $aJ_2 + aJ_4$. For y in $I_2 \cup I_4$ we have $0 \leq y \leq 8as_n^2 < \frac{s_n}{c_n}$, so that $0 \leq \frac{yc_n}{s_n} < 1$. Again setting $y = \varepsilon s_n$ we have, by Theorem 1 (3),

$$q(y) < \exp[-\frac{y^2}{2s_n^2}(1 - \frac{yc_n}{2s_n})].$$

Since $y < 8as_n^2$, we have from (7) and (11)

$$\frac{yc_n}{2s_n} < 4as_nc_n < 8\omega \leq \frac{\delta^2}{8}.$$

Hence for y in $I_2 \cup I_4$ it follows that

$$q(y) < \exp[-\frac{y^2}{2s_n^2}(1 - \frac{\delta^2}{8})],$$

$$\exp(ay)q(y) < \exp[ay - \frac{y^2}{2s_n^2}(1 - \frac{\delta^2}{8})],$$

and hence that

$$aJ_2 + aJ_4 < a\int_0^{as_n^2(1 - \delta)} \exp[ay - \frac{y^2}{2s_n^2}(1 - \frac{\delta^2}{8})]dy$$

$$+ a\int_{as_n^2(1 + \delta)}^{8as_n^2} \exp[ay - \frac{y^2}{2s_n^2}(1 - \frac{\delta^2}{8})]dy.$$

Let $f(y) = ay - \frac{y^2}{2s_n^2}(1 - \frac{\delta^2}{8})$. Then $f(y)$ is a quadratic in y and takes on its maximum value when

$$a - \frac{y}{s_n^2}(1 - \frac{\delta^2}{8}) = 0,$$

or, at the point

$$y_o = \frac{as_n^2}{1 - \frac{\delta^2}{8}} .$$

Clearly $as_n^2(1 - \delta) < y_o$. Since $0 < 1 - \frac{\delta}{8} - \frac{\delta^2}{8}$ for $0 < \delta < \frac{1}{8}$ it follows that $1 < 1 + \delta - \frac{\delta^2}{8} - \frac{\delta^3}{8} = (1 + \delta)(1 - \frac{\delta^2}{8})$, and hence

$$\frac{1}{1 - \frac{\delta^2}{8}} < 1 + \delta$$

Consequently

$$y_o = \frac{as_n^2}{1 - \frac{\delta^2}{8}} < as_n^2(1 + \delta),$$

and thus y_o is in I_3 . Furthermore, $as_n^2 < y_o$, and hence f is maximized on $I_2 \cup I_4$ at the point $as_n^2(1 + \delta)$. See Figure 1. Hence for all y in $I_2 \cup I_4$,

$$f(y) \leq f(as_n^2(1 + \delta)) = a^2s_n^2(1 + \delta) - \frac{a^2s_n^2(1 + \delta)^2}{2}(1 - \frac{\delta^2}{8})$$

$$= \frac{a^2s_n^2}{2}[2 + 2\delta - (1 + 2\delta + \delta^2) + \frac{\delta^2}{8}(1 + \delta)^2]$$

$$= \frac{a^2s_n^2}{2}[1 - \delta^2 + \frac{\delta^2}{8}(1 + \delta)^2]$$

$$< \frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{8}) ,$$

since $\frac{(1 + \delta)^2}{8} < \frac{1}{2}$ for $0 < \delta < \frac{1}{8}$. Thus from (18) we have

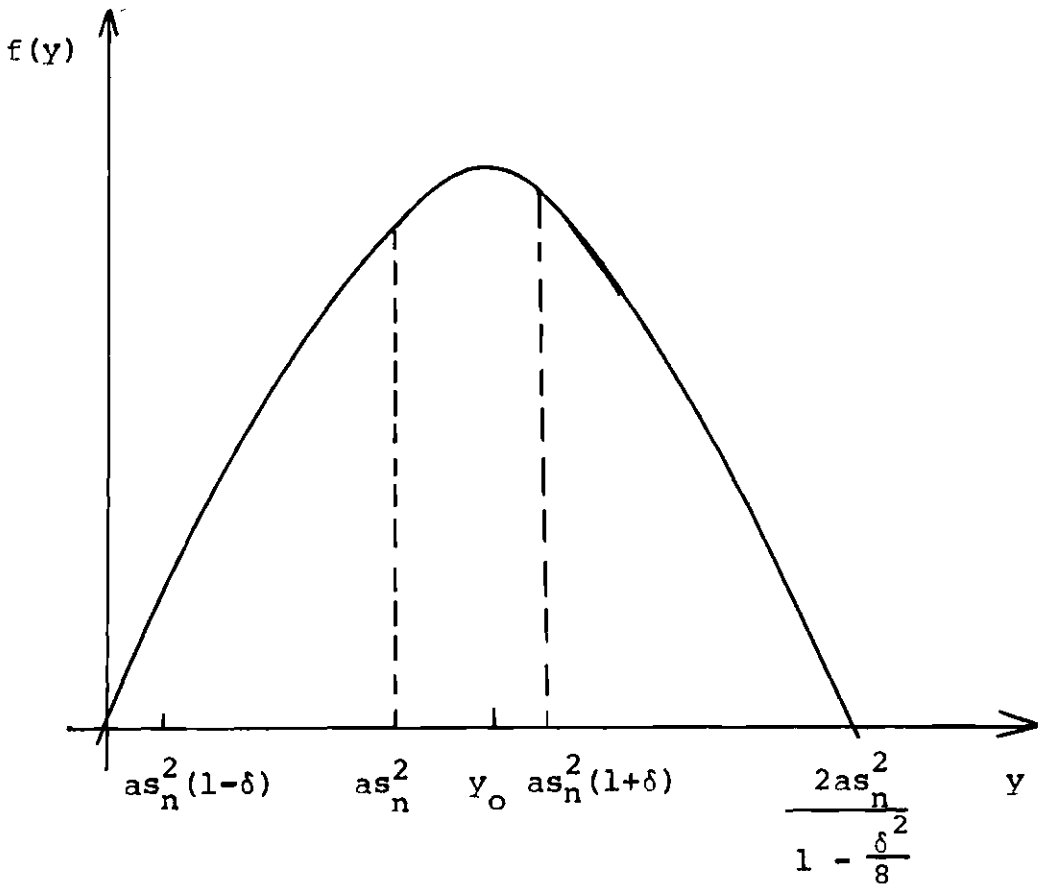

Figure 1. The Graph of $f(y) = ay - \frac{y^2}{2s_n^2}(1 - \frac{\delta^2}{8})$.

$$aJ_2 + aJ_4 < a\int_0^{as_n^2(1-\delta)} \exp[\frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{2})]dy$$

$$+ \ a\int_{as_n^2(1+\delta)}^{8as_n^2} \exp[\frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{2})]dy$$

$$< a\int_0^{8as_n^2} \exp[\frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{2})]dy$$

$$= 8a^2s_n^2 \exp[\frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{2})].$$

Since (7) states that $\delta^2 \geq 16 \frac{\log \lambda}{\lambda}$, it follows that $2\log \lambda \leq \frac{\lambda\delta^2}{8}$. Since $\log 32\lambda = \log 32 + \log \lambda < 2\log \lambda$, we have $\log 32\lambda < \frac{\lambda\delta^2}{8}$, or, $\frac{\log 32\lambda}{\lambda} < \frac{\delta^2}{8}$. Let $h(\lambda) = \frac{\log 32\lambda}{\lambda}$. Then $h'(\lambda) = \frac{1 - \log 32\lambda}{\lambda^2}$, so that $h'(\lambda) < 0$ if $\lambda > \frac{e}{32}$. Hence $h(\lambda)$ is decreasing for $\lambda > 2^{14}$. Thus it follows from (12) that

$$\frac{\log 32a^2s_n^2}{a^2s_n^2} < \frac{\delta^2}{8}$$

or,

$$\log 32a^2s_n^2 < \frac{a^2s_n^2\delta^2}{8}.$$

Then

$$\exp[\log(32a^2s_n^2)] < \exp\frac{a^2s_n^2\delta^2}{8},$$

$$\exp(-\frac{a^2s_n^2\delta^2}{8}) < \frac{1}{32a^2s_n^2}. \qquad (19)$$

Hence

$$aJ_2 + aJ_4 < 8a^2s_n^2 \exp[\frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{2})] \qquad (20)$$

$$= 8a^2s_n^2 \exp[\frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{4}) - \frac{a^2s_n^2\delta^2}{8}]$$

$$= 8a^2s_n^2 \exp[\frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{4})]\exp(-\frac{a^2s_n^2\delta^2}{8})$$

$$< \frac{1}{4}\exp[\frac{a^2s_n^2}{2}(1 - \frac{\delta^2}{4})]$$

$$< \frac{1}{4}E[\exp(aS_n)],$$

the last two inequalities following from (19) and (13) respectively. From (13), (14), (17), and (20) we have

$$aJ_3 > \frac{1}{2}E[\exp(aS_n)] \qquad (21)$$

$$> \frac{1}{2} \exp[\frac{a^2 s_n^2}{2}(1 - \frac{\delta^2}{4})]$$

$$> \frac{1}{2} \exp[\frac{a^2 s_n^2}{2}(1 - \delta)]$$

since $\delta > \frac{\delta^2}{4}$. On the other hand, since $q(y)$ is monotone decreasing and since (9) holds,

$$aJ_3 = a\int_{as_n^2(1-\delta)}^{as_n^2(1+\delta)} \exp(ay)q(y)dy$$

$$< a \exp[a^2 s_n^2(1 + \delta)]q(x)as_n^2(1 + \delta)$$

$$< 2a^2 s_n^2 \exp[a^2 s_n^2(1 + \delta)]q(x).$$

It follows from (21) that

$$q(y) > \frac{aJ_3}{2a^2 s_n^2 \exp[a^2 s_n^2(1 + \delta)]} \tag{22}$$

$$> \frac{\frac{1}{2} \exp[\frac{a^2 s_n^2}{2}(1 - \delta)}{2a^2 s_n^2 \exp[a^2 s_n^2(1 + \delta)]}$$

$$= \frac{1}{4a^2 s_n^2} \exp[- \frac{a^2 s_n^2}{2}(1 + 3\delta)]$$

$$= \frac{1}{4a^2s_n^2} \exp[-\frac{a^2s_n^2}{2}(1 + 4\delta) + \frac{a^2s_n^2\delta}{2}]$$

$$= \frac{1}{4a^2s_n^2} \exp[-\frac{a^2s_n^2}{2}(1 + 4\delta)]\exp[\frac{a^2s_n^2\delta}{2}] \ .$$

Now $\log 4\lambda < 2\log \lambda$, and from (7) it follows that

$$\frac{2\log \lambda}{\lambda} \leq \frac{\delta^2}{8} < \frac{\delta^2}{4} < \frac{\delta}{2} .$$

Thus $\frac{\log 4\lambda}{\lambda} < \frac{\delta}{2}$. Since $\frac{\log 4\lambda}{\lambda}$ is decreasing for $\lambda > 2^{14}$, and since $a^2s_n^2 > \lambda$ from (12), it follows that

$$\log(4a^2s_n^2) < \frac{a^2s_n^2\delta}{2} \quad ,$$

or,

$$\exp[\frac{a^2s_n^2\delta}{2}] > 4a^2s_n^2 \ .$$

Hence from (22) we have

$$q(x) > \exp[-\frac{a^2s_n^2}{2}(1 + 4\delta)] \ ,$$

and from (9) we obtain

$$q(x) > \exp[-\frac{x^2}{2s_n^2(1-\delta)^2}(1+4\delta)] \ . \tag{23}$$

For $0 < \delta < \frac{1}{8}$, we have

$$0 < 2 - 15\delta + 8\delta^2 \ ,$$

$$1 + 4\delta < 1 + 6\delta - 15\delta^2 + 8\delta^3 = (1+8\delta)(1-\delta)^2 \ ,$$

$$\frac{1+4\delta}{(1-\delta)^2} < (1+8\delta) \ .$$

Thus from (23) it follows that

$$q(x) > \exp[-\frac{x^2}{2s_n^2}(1+8\delta)] = \exp[-\frac{x^2}{2s_n^2}(1+\varepsilon)] \ .$$

Since $q(x) = P[S_n > x]$, we have

$$P[S_n > x] > \exp[-\frac{x^2}{2s_n^2}(1+\varepsilon)] \ .$$

This completes the proof of Theorem 2.

<u>Lemma 4</u>

Let X be a random variable with finite second moment. Let σ^2 denote the variance of X, and let $\tilde{x}$ denote any median of X. Then

$$|\tilde{x} - E(X)| \leq \sqrt{2}\sigma \ .$$

Proof. By Chebyshev's inequality, for every $\delta > 0$,

$$P[|X - E(X)| \leq \sqrt{2 + \delta}\ \sigma] = 1 - P[|X - E(X)| > \sqrt{2 + \delta}\ \sigma]$$

$$\geq 1 - \frac{\sigma^2}{(2 + \delta)\sigma^2}$$

$$= 1 - \frac{1}{2 + \delta}$$

$$> \frac{1}{2} \ .$$

Since

$$[|X - E(X)| \leq \sqrt{2 + \delta}\ \sigma]$$

$$= [X \geq E(X) - \sqrt{2 + \delta}\ \sigma] \cap [X \leq E(X) + \sqrt{2 + \delta}\ \sigma] \ ,$$

we have

$$P[X \geq E(X) - \sqrt{2 + \delta}\ \sigma] \geq P[|X - E(X)| \leq \sqrt{2 + \delta}\ \sigma] > \frac{1}{2}$$

and

$$P[X \leq E(X) + \sqrt{2 + \delta}\ \sigma] \geq P[|X - E(X)| \leq \sqrt{2 + \delta}\ \sigma] > \frac{1}{2} \ .$$

Hence, if $\tilde{x}$ is a median of X, we must have

$$E(X) - \sqrt{2 + \delta}\,\sigma \leq \tilde{x} \leq E(X) + \sqrt{2 + \delta}\,\sigma ,$$

or,

$$|\tilde{x} - E(X)| \leq \sqrt{2 + \delta}\,\sigma ,$$

from the definition of a median of X. Since $\delta > 0$ is arbitrary, we have

$$|\tilde{x} - E(X)| \leq \sqrt{2}\sigma .$$

Lemma 5 (P. Levy's Inequalities)

Let $X_1, \ldots, X_n$ be n independent random variables, and let $S_k = X_1 + \ldots + X_k$. Let $\hat{s}_j$ denote a median of $S_j - S_n$. Then, if $\varepsilon > 0$ is any constant, we have

$$P[\max_{1\leq k\leq n} (S_k - \hat{s}_k) \geq \varepsilon] \leq 2P[S_n \geq \varepsilon] \tag{24}$$

and

$$P[\max_{1\leq k\leq n} |S_k - \hat{s}_k| \geq \varepsilon] \leq 2P[|S_n| \geq \varepsilon] . \tag{25}$$

Proof. To prove (24), let $S_o = 0$, $S_k^* = \max_{1\leq k\leq n} (S_j - \hat{s}_j)$,

$1 \le k \le n$, $A_k = [S_{k-1} < \varepsilon] \cap [S_k - \mathfrak{m}_k \ge \varepsilon]$, and $B_k = [S_n - S_k + \mathfrak{m}_k \ge 0]$. Then $\{A_1, \ldots, A_n\}$ are disjoint, and $\bigcup_{k=1}^{n} A_k = [S_n^* \ge \varepsilon]$. But for every k, $1 \le k \le n$,

$$B_k \cap A_k = [S_n - S_k + \mathfrak{m}_k \ge 0] \cap [S_{k-1} \le \varepsilon] \cap [S_k - \mathfrak{m}_k \ge \varepsilon]$$

$$\subset [S_n \ge S_k - \mathfrak{m}_k] \cap [S_k - \mathfrak{m}_k \ge \varepsilon] \subset [S_n \ge \varepsilon].$$

Hence

$$\bigcup_{k=1}^{n} (A_k \cap B_k) \subset [S_n \ge \varepsilon].$$

Since $\{A_1, \ldots, A_n\}$ are disjoint, so are $\{A_1 \cap B_1, \ldots, A_n \cap B_n\}$. Note that $-\mathfrak{m}_k$ is a median of $S_n - S_k$, so $P(B_k) \ge \frac{1}{2}$. Finally, A_k and B_k are independent[1] for every k, $1 \le k \le n$.

Hence

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[1] Let $X_1, \ldots, X_n$ be n independent random variables. Let $1 \le n_1 < n_2 < \ldots < n_k = n$, and let f_1 be a Borel measurable function of n_1 variables, f_2 a Borel measurable function of $n_2 - n_1$ variables, ..., f_k a Borel measurable function of $n_k - n_{k-1}$ variables. Then the k random variables $f_1(X_1, \ldots, X_{n_1})$, $f_2(X_{n_1+1}, \ldots, X_{n_2})$, ..., $f_k(X_{n_{k-1}+1}, \ldots, X_{n_k})$ are independent.

$$P[S_n \geq \varepsilon] \geq P[\bigcup_{k=1}^{n} (A_k \cap B_k)] = \sum_{k=1}^{n} P(A_k \cap B_k) = \sum_{k=1}^{n} P(A_k)P(B_k)$$

$$\geq \frac{1}{2} \sum_{k=1}^{n} P(A_k) = \frac{1}{2} P(\bigcup_{k=1}^{n} A_k) = \frac{1}{2} P[S_n^* \geq \varepsilon].$$

This proves (24). To prove (25), let $S_o = 0$, $S_k^{**} = \min\limits_{1 \leq j \leq k} (S_j - \hat{s}_j)$, $1 \leq k \leq n$, $C_k = [S_{k-1}^{**} > -\varepsilon] \cap [S_k - \hat{s}_k \leq -\varepsilon]$, and let $D_k =$ $[S_n - S_k + \hat{s}_k \leq 0]$. Then $\{C_1, \ldots, C_n\}$ are disjoint, and $\bigcup\limits_{k=1}^{n} C_k = [S_n^{**} \leq -\varepsilon]$. For every k, $1 \leq k \leq n$,

$$C_k \cap D_k \subset [S_n \leq S_k - \hat{s}_k] \cap [S_k - \hat{s}_k \leq -\varepsilon] \subset [S_n \leq -\varepsilon].$$

Thus

$$\bigcup_{k=1}^{n} (C_k \cap D_k) \subset [S_n \leq -\varepsilon].$$

Since $\{C_1, \ldots, C_n\}$ are disjoint, so are $\{C_1 \cap D_1, \ldots, C_n \cap D_n\}$. Also, $P(D_k) \geq \frac{1}{2}$, since $-\hat{s}_k$ is a median of $S_n - S_k$. Finally, C_k and D_k are independent for every k, $1 \leq k \leq n$. Hence

$$P[S_n \leq -\varepsilon] \geq P[\bigcup_{k=1}^{n} (C_k \cap D_k)] = \sum_{k=1}^{n} P(C_k \cap D_k) = \sum_{k=1}^{n} P(C_k)P(D_k)$$

$$\geq \frac{1}{2} \sum_{k=1}^{n} P(C_k) = \frac{1}{2} P(\bigcup_{k=1}^{n} C_k) = \frac{1}{2} P[S_n^{**} \leq -\varepsilon].$$

Combining this result with (24), we obtain

$$P[\min_{1\leq k\leq n}(S_k - \hat{S}_k) \leq -\varepsilon] \leq 2P[S_n \leq -\varepsilon]$$

and

$$P[\max_{1\leq k\leq n}(S_k - \hat{S}_k) \geq \varepsilon] \leq 2P[S_n \geq \varepsilon].$$

Now

$$[\max_{1\leq k\leq n}|S_k - \hat{S}_k| \geq \varepsilon]$$

$$= [\max_{1\leq k\leq n}(S_k - \hat{S}_k) \geq \varepsilon] \cup [\min_{1\leq k\leq n}(S_k - \hat{S}_k) \leq -\varepsilon]$$

and

$$[|S_n| \geq \varepsilon] = [S_n \geq \varepsilon] \cup [S_n \leq -\varepsilon],$$

where each union is a union of disjoint sets. Hence

$$P[\max_{1\leq k\leq n}|S_k - \hat{S}_k| \geq \varepsilon]$$

$$= P\{[\max_{1\leq k\leq n}(S_k - \hat{S}_k) \geq \varepsilon] \cup [\min_{1\leq k\leq n}(S_k - \hat{S}_k) \leq -\varepsilon]\}$$

$$= P[\max_{1\leq k\leq n}(S_k - \hat{S}_k) \geq \varepsilon] + P[\min_{1\leq k\leq n}(S_k - \hat{S}_k) \leq -\varepsilon]$$

$$\leq 2\{P[S_n \geq \varepsilon] + P[S_n \leq -\varepsilon]\}$$

$$= 2P\{[S_n \geq \varepsilon] \cup [S_n \leq -\varepsilon]\}$$

$$= 2P[|S_n| \geq \varepsilon].$$

This proves (25).

<u>Lemma</u> <u>6</u>

Let $X_1, \ldots, X_n$ be n independent random variables with finite second moments such that $E(X_k) = 0$ for $1 \leq k \leq n$. Let $S_k = X_1 + \ldots + X_k$, and let s_n^2 denote the variance of S_n. Then

$$P[\max_{1 \leq k \leq n} S_k \geq \varepsilon] \leq 2P[S_n \geq \varepsilon - \sqrt{2}s_n].$$

Proof. Let s_k^2 denote the variance of $S_k - S_n$, and let $\hat{s}_k$ denote a median of $S_k - S_n$. Then by Lemma 4 we have

$$|\hat{s}_k| \leq \sqrt{2}s_k \leq \sqrt{2}s_n ,$$

since $E(S_k - S_n) = 0$. Hence

$$-\hat{s}_k \geq -\sqrt{2}s_n .$$

Thus for every k, $1 \leq k \leq n$,

$$[S_k \geq \varepsilon] \subset [S_k - \hat{s}_k \geq \varepsilon - \sqrt{2}s_n] ,$$

$$\bigcup_{k=1}^{n}[S_k \geq \varepsilon] \subset \bigcup_{k=1}^{n}[S_k - \hat{S}_k \geq \varepsilon - \sqrt{2}s_n].$$

Hence

$$\begin{aligned} P[\max_{1\leq k\leq n} S_k \geq \varepsilon] &= P(\bigcup_{k=1}^{n}[S_k \geq \varepsilon]) \\ &\leq P(\bigcup_{k=1}^{n}[S_k - \hat{S}_k \geq \varepsilon - \sqrt{2}s_n]) \\ &= P[\max_{1\leq k\leq n}(S_k - \hat{S}_k) \geq \varepsilon - \sqrt{2}s_n] \\ &\leq 2P[S_n \geq \varepsilon - \sqrt{2}s_n] \ , \end{aligned}$$

by conclusion (24) of Lemma 5.

<u>Theorem 3</u> (Kolmogorov's Law of the Iterated Logarithm)

Let $\{X_n\}$ be a sequence of independent random variables such that $d_n = \text{ess sup } |X_n| < \infty$ and $E(X_n) = 0$ for all n. Let $S_n = X_1 + \ldots + X_n$, and suppose that $s_n^2 = E(S_n^2) \to \infty$ and $d_n = o\left(\dfrac{s_n}{\sqrt{\log \log s_n^2}}\right)$ as $n \to \infty$. Then

$$P[\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log \log s_n^2}} = 1] = 1.$$

Equivalently, if $\delta > 0$, then

with probability one only finitely many of the events (26)

$$S_n > (1 + \delta)\sqrt{2s_n^2 \log \log s_n^2} \quad \text{occur}$$

and

with probability one infinitely many of the events (27)

$$S_n > (1 - \delta)\sqrt{2s_n^2 \log \log s_n^2} \quad \text{occur} .$$

Proof. We first prove (26). Let $D_n = \max\{d_k : 1 \leq k \leq n\}$, $U_n = \max\{S_k : 1 \leq k \leq n\}$, and let x_n^2 denote the variance of X_n. Let $0 < \delta < \frac{1}{2}$. Since $s_n^2 \to \infty$ as $n \to \infty$, there is a positive integer N_1 such that for all $n \geq N_1$,

$$s_n^2 > e^e \tag{28}$$

and

$$\sqrt{\log \log s_n^2} > \frac{4}{\delta} . \tag{29}$$

Since $d_n = o\left(\frac{s_n}{\sqrt{\log \log s_n^2}}\right)$, there is a positive integer N_2 such that for all $n \geq N_2$,

$$\frac{D_n^2}{s_n^2} < \frac{\delta}{16} \tag{30}$$

and

$$D_n \sqrt{\frac{\log \log s_n^2}{s_n^2}} < \frac{\delta}{4} . \tag{31}$$

Since $x_n^2 \leq d_n^2 = o\left(\dfrac{s_n^2}{\log\log s_n^2}\right)$, we have

$$\frac{s_{n+1}^2}{s_n^2} = 1 + \frac{x_{n+1}^2}{s_n^2} \leq 1 + \frac{d_{n+1}^2}{s_n^2} .$$

For some positive integer P, we have for all $n \geq P$,

$$1 + \frac{d_{n+1}^2}{s_n^2} \leq 1 + \frac{s_{n+1}^2}{s_n^2} \cdot \frac{1}{\log\log s_{n+1}^2} .$$

Thus

$$\frac{s_{n+1}^2}{s_n^2}\left(1 - \frac{1}{\log\log s_{n+1}^2}\right) \leq 1 \quad \text{for all } n \geq P.$$

From this it follows that $\lim\limits_{n \to \infty} \dfrac{s_{n+1}^2}{s_n^2} \leq 1$, since $\log\log s_{n+1}^2 \to \infty$ as $n \to \infty$. Since $s_{n+1}^2 \geq s_n^2$ for all n, we have $1 \leq \dfrac{s_{n+1}^2}{s_n^2}$.

Hence, $\dfrac{s_{n+1}^2}{s_n^2} \to 1$ as $n \to \infty$. Thus there is a positive integer N_3 such that

$$\frac{s_{n+1}^2}{s_n^2} \leq 1 + \frac{\delta}{4} \quad \text{if } n \geq N_3 .$$

Let

$$n_o = \max\{N_1, N_2, N_3\} \ . \tag{32}$$

Then

$$\frac{s^2_{n_o+1}}{s^2_{n_o}} \leq 1 + \frac{\delta}{4} \ .$$

Since $s^2_n \to \infty$ as $n \to \infty$, there is a positive integer q such that

$$\frac{s^2_q}{s^2_{n_o}} > 1 + \frac{\delta}{4} \ .$$

Let r be the smallest such integer, and let $n_1 = r - 1$. Since $r > n_o + 1$, we have $n_1 > n_o$. We now have

$$\frac{s^2_{n_1+1}}{s^2_{n_o}} > 1 + \frac{\delta}{4}$$

and

$$\frac{s^2_{n_1}}{s^2_{n_o}} \leq 1 + \frac{\delta}{4} \ ,$$

since $r = n_1 + 1$ was the smallest integer q such that

$\frac{s_q^2}{s_{n_o}^2} > 1 + \frac{\delta}{4}$. Proceeding in a similar fashion, we define

inductively a strictly increasing sequence of integers n_o, n_1, ..., n_{k-1}, where n_k is chosen so that

$$\frac{s_{n_k}^2}{s_{n_{k-1}}^2} \le 1 + \frac{\delta}{4} \tag{33}$$

and

$$\frac{s_{n_k+1}^2}{s_{n_{k-1}}^2} > 1 + \frac{\delta}{4} . \tag{34}$$

Now for every j, $1 \le j \le k$,

$$\frac{s_{n_j}^2}{s_{n_{j-1}}^2} = \frac{s_{n_j+1}^2 - x_{n_j+1}^2}{s_{n_{j-1}}^2} .$$

Since $x_{n_j+1}^2 \le d_{n_j+1}^2 \le D_{n_j+1}^2$, we have

$$\frac{s_{n_j+1}^2 - x_{n_j+1}^2}{s_{n_{j-1}}^2} \ge \frac{s_{n_j+1}^2 - D_{n_j+1}^2}{s_{n_{j-1}}^2} > (1 + \delta)\left(1 - \frac{D_{n_j+1}^2}{s_{n_j+1}^2}\right)$$

by (34). From (30) and (32) we obtain

$$(1 + \delta)\left(1 - \frac{D^2_{n_j+1}}{s^2_{n_j+1}}\right) > (1 + \frac{\delta}{4})(1 - \frac{\delta}{16}) .$$

Since $0 < \delta < \frac{1}{2}$,

$$(1 + \frac{\delta}{4})(1 - \frac{\delta}{16}) = (1 + \frac{\delta}{4}) - \frac{\delta}{8}(\frac{1}{2} + \frac{\delta}{8}) > 1 + \frac{\delta}{8} .$$

Hence, for every j, $1 \leq j \leq k$,

$$\frac{s^2_{n_j}}{s^2_{n_{j-1}}} > 1 + \frac{\delta}{8} .$$

Thus

$$s^2_{n_k} = \frac{s^2_{n_k}}{s^2_{n_{k-1}}} \cdot \frac{s^2_{n_{k-1}}}{s^2_{n_{k-2}}} \cdot \ldots \cdot \frac{s^2_{n_2}}{s^2_{n_1}} \cdot s^2_{n_1}$$

$$> (1 + \frac{\delta}{8})^{k-1} s^2_{n_1}$$

$$> (1 + \frac{\delta}{8})^{k}$$

by (28) and (32). Consequently

$$s_{n_k}^2 > (1 + \frac{\delta}{8})^k \, . \tag{35}$$

Let $F(v) = \sqrt{2v \log \log v}$. From (33) we have

$$\frac{F(s_{n_k}^2)}{F(s_{n_{k-1}}^2)} < 1 + \frac{\delta}{4} \, . \tag{36}$$

If $S_n > F(s_n^2)(1 + \delta)$ for some n, $n_{k-1} \leq n \leq n_k$, then $U_{n_k} > F(s_{n_{k-1}}^2)(1 + \delta)$. By the Borel-Cantelli Lemma, (26) will be proved when it is shown that the series

$$\sum_{k=1}^{\infty} P[U_{n_k} > F(s_{n_{k-1}}^2)(1 + \delta)]$$

converges. Since $0 < \delta < \frac{1}{2}$, it follows from (36) that

$$P[U_{n_k} > F(s_{n_{k-1}}^2)(1 + \delta)] \leq P[U_{n_k} > F(s_{n_k}^2)(1 + \frac{\delta}{2})]$$

and from Lemma 6,

$$P[U_{n_k} > F(s_{n_k}^2)(1 + \frac{\delta}{2})] \leq 2P[S_n > F(s_{n_k}^2)(1 + \frac{\delta}{2}) - \sqrt{2}s_{n_k}] \, .$$

Since $\sqrt{2}s_{n_k} < \frac{\delta}{4} F(s_{n_k}^2)$ by (29) and (32), we have

$$P[U_{n_k} > F(s^2_{n_{k-1}})(1 + \delta)] < 2P[S_{n_k} > F(s^2_{n_k})(1 + \frac{\delta}{4})].$$

Now let $\varepsilon = \sqrt{2\log\log s^2_{n_k}}\,(1 + \frac{\delta}{4})$, $c_n = \dfrac{D_{n_k}}{s_{n_k}}$. From (31) and (32) it follows that

$$\frac{D_{n_k}}{s_{n_k}} \sqrt{2\log\log s^2_{n_k}} < \sqrt{2}\,\frac{\delta}{4} < \frac{\delta}{2}$$

and hence that

$$\varepsilon c_n < (1 + \frac{\delta}{4}) \cdot \frac{\delta}{2} < 1.$$

Hence by Theorem 1,

$$2P[S_{n_k} > F(s^2_{n_k})(1 + \frac{\delta}{4})] < 2\exp[-(\log\log s^2_{n_k})(1 + \frac{\delta}{4})^2(1 - \theta)],$$

where $\theta = D_{n_k}\sqrt{\dfrac{\log\log s^2_{n_k}}{s^2_{n_k}}}\,(1 + \frac{\delta}{4})(\frac{\sqrt{2}}{2}) < (1 + \frac{\delta}{4}) \cdot \frac{\delta}{4}$.

Thus

$$(1 + \frac{\delta}{4})^2(1 - \theta) > (1 + \frac{\delta}{4})^2(1 - \frac{\delta}{4} - \frac{\delta^2}{16})$$

$$= 1 + \frac{\delta}{4} - \frac{\delta}{8}(\delta + \frac{3\delta^2}{16} + \frac{\delta^3}{64}) > 1 + \frac{\delta}{8},$$

since $0 < \delta < \frac{1}{2}$. Hence

$$2\exp[(-\log\log s_{n_k}^2)(1+\tfrac{\delta}{4})^2(1-\theta)]$$

$$< 2\exp[(-\log\log s_{n_k}^2)(1+\tfrac{\delta}{8})].$$

By (35)

$$2\exp[(-\log\log s_{n_k}^2)(1+\tfrac{\delta}{8})] < 2[\exp\{\log\log(1+\tfrac{\delta}{8})^k\}]^{-(1+\frac{\delta}{8})}$$

$$= \frac{2}{k^{(1+\frac{\delta}{8})}[\log(1+\frac{\delta}{8})]^{(1+\frac{\delta}{8})}}.$$

Since $\sum_{k=1}^{n} \frac{1}{k^{(1+\frac{\delta}{8})}}$ converges, it follows that the series

$$\sum_{k=1}^{\infty} P[U_{n_k} > F(s_{n_{k-1}}^2)(1+\delta)]$$

converges. This proves (26). To prove (27), let $0 < \delta < \frac{1}{2}$, and let $\eta > 0$ be given. Since $s_n^2 \to \infty$ as $n \to \infty$ and $d_n = o\left(\frac{s_n}{\sqrt{\log\log s_n^2}}\right)$, there is a positive integer N_1 such that

$$s_n^2 > 2e^e \quad \text{for all } n \geq N_1 \tag{37}$$

and

$$\frac{D_n^2}{s_n^2} < \frac{1}{2} \quad \text{for all } n \geq N_1. \tag{38}$$

In the proof of (26) it was shown that for n sufficiently large, the probability that at least one of the inequalities

$$S_k > (1 + \delta)F(s_k^2) \qquad (k = n, n + 1, \ldots, n + p)$$

holds is smaller than $\frac{\eta}{4}$ for every positive integer p. Applying the same proof to $-S_k$, it follows that for n sufficiently large, the probability that at least one of the inequalities

$$-S_k > (1 + \delta)F(s_k^2) \qquad (k = n, n + 1, \ldots, n + p)$$

holds is smaller than $\frac{\eta}{4}$ for every positive integer p. Hence the probability that at least one of the inequalities

$$|S_k| > (1 + \delta)F(s_k^2) \qquad (k = n, n + 1, \ldots, n + p)$$

holds is smaller than $\frac{\eta}{2}$. Thus there exists a positive integer N_2 such that the probability that all of the inequalities

$$|S_k| \leq (1 + \delta)F(s_k^2) \tag{39}$$

$$< 2F(s_k^2) \qquad (k = N_2, N_2 + 1, \ldots, N_2 + p)$$

hold is greater than $1 - \frac{\eta}{2}$ for every positive integer p. Let

$$n_o = \max\{N_1, N_2\} ,$$

and define inductively a strictly increasing sequence of positive integers $n_o, n_1, \ldots, n_{k-1}$, where n_k is chosen so that

$$\frac{s^2_{n_k-1}}{s^2_{n_{k-1}}} \le \frac{16}{\delta^2} < \frac{s^2_{n_k}}{s^2_{n_{k-1}}} . \tag{40}$$

Since $s^2_{n_k-1} \ge s^2_{n_k} - D^2_{n_k}$, it follows that

$$\frac{s^2_{n_k}}{s^2_{n_{k-1}}} \le \frac{s^2_{n_k-1} + D^2_{n_k}}{s^2_{n_{k-1}}} = \frac{s^2_{n_k-1}}{s^2_{n_{k-1}}} \left(1 + \frac{D^2_{n_k}}{s^2_{n_k-1}} \right)$$

$$\le \frac{16}{\delta^2} \left[1 + \frac{D^2_{n_k}}{s^2_{n_k} - D^2_{n_k}} \right] = \frac{16}{\delta^2} \left(1 + \frac{1}{\frac{s^2_{n_k}}{D^2_{n_k}} - 1} \right) .$$

It now follows from (38) that

$$\frac{s^2_{n_k}}{s^2_{n_{k-1}}} < \frac{32}{\delta^2} ,$$

$$s_{n_k}^2 < s_{n_o}^2 \left(\frac{32}{\delta^2}\right)^k < \left(s_{n_o}^2 \cdot \frac{32}{\delta^2}\right)^k . \tag{41}$$

Let $T_k = S_{n_k} - S_{n_{k-1}}$, $t_k^2 = E(T_k^2) = s_{n_k}^2 - s_{n_{k-1}}^2$. From (40) we have

$$s_{n_{k-1}}^2 < \frac{\delta^2}{16} \cdot s_{n_k}^2 < \frac{\delta}{4} s_{n_k}^2 ,$$

$$t_k^2 = s_{n_k}^2 - s_{n_{k-1}}^2 > s_{n_k}^2 \left(1 - \frac{\delta}{4}\right) > \frac{1}{2} s_{n_k}^2 . \tag{42}$$

Consider the function $g(x) = x \log a - a \log x$. If $e < a < x$, then $g(x) \geq 0$, since $g(a) = 0$ and $g'(x) = \log a - \frac{a}{x} > 0$. Let $a = t_k^2$, $x = s_{n_k}^2$. Then because of (37) and (42)

$$\frac{2\log\log t_k^2}{2\log\log s_{n_k}^2} > \frac{\log\log t_k^2}{\log\log s_{n_k}^2} > \frac{t_k^2}{s_{n_k}^2} > \left(1 - \frac{\delta}{4}\right)$$

and it follows that

$$\frac{F(t_k^2)}{F(s_{n_k}^2)} > 1 - \frac{\delta}{4} .$$

Since $\left(1 - \frac{\delta}{4}\right)^2 = 1 - \frac{\delta}{2} + \frac{\delta^2}{16} > 1 - \frac{\delta}{2}$, we have

$$F(t_k^2)(1 - \frac{\delta}{4}) > F(s_{n_k}^2)(1 - \frac{\delta}{2}). \tag{43}$$

On the other hand it also follows from (40) that

$$s_{n_{k-1}} < \frac{\delta}{4} s_{n_k},$$

$$F(s_{n_{k-1}}^2) < \frac{\delta}{4} F(s_{n_k}^2),$$

$$F(s_{n_k}^2)(1 - \frac{\delta}{2}) - 2F(s_{n_{k-1}}^2) > F(s_{n_k}^2)(1 - \frac{\delta}{2}) - \frac{\delta}{2} F(s_{n_k}^2)$$

$$= (1 - \delta)F(s_{n_k}^2) .$$

Consider the event $V_k = [T_k > F(t_k^2)(1 - \frac{\delta}{4})]$. We will now show that with probability one infinitely many of the events V_k occur. First note that the random variables

$$\{S_{n_k} - S_{n_{k-1}} : k = 1, 2, \ldots\}$$

are independent, and hence the events $\{V_k\}$ are independent. By the Borel-Cantelli Lemma it is sufficient to prove that $\sum_{k=1}^{\infty} P(V_k) = \infty$. To estimate $P(V_k)$ we turn to Theorem 2, where we set $s_n^2 = t_k^2$, $c_n = D_{n_k}$, $x = F(t_k^2)(1 - \frac{\delta}{4})$. Then

$$\frac{xc_n}{s_n^2} = \frac{F(t_k^2)(1-\frac{\delta}{4})D_{n_k}}{t_k^2} < \frac{F(t_k^2)D_{n_k}}{t_k^2} < \frac{F(s_{n_k}^2)D_{n_k}}{t_k^2} \quad .$$

From (42)

$$\frac{F(s_{n_k}^2)D_{n_k}}{t_k^2} < \frac{2F(s_{n_k}^2)D_{n_k}}{s_{n_k}^2} = 2\sqrt{2}\; D_{n_k}\sqrt{\frac{\log\log s_{n_k}^2}{s_{n_k}^2}} \to 0$$

as $k \to \infty$. Hence $\dfrac{xc_n}{s_n^2} \to 0$ as $k \to \infty$. Also

$$\frac{x^2}{s_n^2} = \frac{F^2(t_k^2)}{t_k^2}(1-\frac{\delta}{4})^2 = 2(\log\log t_k^2)(1-\frac{\delta}{4})^2$$

$$> 2(\log\log\frac{s_{n_k}^2}{2})(1-\frac{\delta}{4})^2 \to \infty \text{ as } k \to \infty.$$

Hence conditions (5) and (6) in Theorem 2 are satisfied for k sufficiently large. Since $\varepsilon \to 0$ as $k \to \infty$ we have for sufficiently large k

$$(1+\varepsilon)(1-\frac{\delta}{4}) < 1.$$

Hence from Theorem 2 we have

$$P[T_k > F(t_k^2)(1 - \tfrac{\delta}{4})] > \exp[-\frac{F^2(t_k^2)}{2t_k^2}(1 - \tfrac{\delta}{4})^2(1 + \varepsilon)]$$

$$> \exp[(-\log\log t_k^2)(1 - \tfrac{\delta}{4})]$$

$$= \exp[\log(\log t_k^2)^{-(1 - \frac{\delta}{4})}]$$

$$= (\log t_k^2)^{-(1 - \frac{\delta}{4})}$$

$$> (\log s_{n_k}^2)^{-(1 - \frac{\delta}{4})}$$

since $t_k^2 < s_{n_k}^2$. We have from (41)

$$\log s_{n_k}^2 < k \log(s_{n_o}^2 \cdot \frac{32}{\delta^2}) .$$

Hence if $c = [\log(s_{n_o}^2 \cdot \frac{32}{\delta^2})]^{-(1 - \frac{\delta}{4})}$, we have

$$P[T_k > F(t_k^2)(1 - \tfrac{\delta}{4})] > \frac{c}{k^{(1 - \frac{\delta}{4})}} ,$$

which proves the divergence of the series $\sum_{k=1}^{\infty} P(V_k)$. Let q be a given positive integer. Let $Y_k = [S_{n_k} > F(s_{n_k}^2)(1 - \delta)]$, and for each positive integer r, let Z_r be the event more than q of the events Y_k, $1 \leq k \leq r$, occur. Let $A_k = [|S_{n_k}| < 2F(s^2_{n_k})]$, and let B be the event all of the events A_k occur. Finally, for each positive integer r, let W_r be the event more than q of the events V_k, $1 \leq k \leq r$, occur. Now suppose $B \cap W_r$ occurs. Then for more than q values of k, $1 \leq k \leq r$, we have

$$S_{n_{k-1}} > -2F(s^2_{n_{k-1}})$$

and

$$T_k > F(t_k^2)(1 - \frac{\delta}{4}) .$$

Hence from (43) and (44) we have

$$S_{n_k} = T_k + S_{n_{k-1}} > F(t_k^2)(1 - \frac{\delta}{4}) - 2F(s^2_{n_k})$$

$$> F(s^2_{n_k})(1 - \frac{\delta}{2}) - 2F(s^2_{n_k}) > F(s^2_{n_k})(1 - \delta).$$

Thus $B \cap W_r \subset Z_r$. It has just been shown that there is a positive integer R such that

$$P(W_r) > 1 - \frac{\eta}{2} \quad \text{if } r \geq R.$$

By the way n_o was chosen,

$$P(B) > 1 - \frac{\eta}{2} .$$

Hence if $r \geq R$,

$$P(B \cap W_r) > 1 - \eta .$$

Since $B \cap W_r \subset Z_r$, it follows that

$$P(Z_r) > 1 - \eta \quad \text{if } r \geq R.$$

Hence with probability one infinitely many of the inequalities

$$S_{n_k} > F(s_{n_k}^2)(1 - \delta)$$

hold. This completes the proof of the theorem.

CHAPTER III

SOME GENERALIZATIONS OF THE LAW

In this chapter some of the generalizations of Kolmogorov's law of the iterated logarithm are outlined. The proofs of the various theorems cited are to be found in Feller [2] and Strassen [8]. Throughout the first part of this chapter, $\{X_k\}$ will be a sequence of random variables, and we let

$$S_n = X_1 + \dots + X_n \tag{1}$$

and

$$V_k(x) = \Pr[X_k \le x]. \tag{2}$$

We shall require that

$$\sigma_k^2 = \int_{-\infty}^{+\infty} x^2 dV_k(x) \quad \text{exists} \quad (k = 1, 2, \dots), \tag{3}$$

$$\int_{-\infty}^{+\infty} x dV_k(x) = 0, \quad (k = 1, 2, \dots) \tag{4}$$

and

$$s_n \to \infty \text{ as } n \to \infty \tag{5}$$

where

$$s_n^2 = \sigma_1^2 + \ldots + \sigma_n^2 \, . \tag{6}$$

We shall also assume that there is a sequence $\lambda_n \downarrow 0$ such that

$$\text{l.u.b.}\ |X_n| < \lambda_n s_n \, . \tag{7}$$

The discussion will be facilitated by adopting the following terminology due to P. Levy :

A sequence $\{\phi_n\}$ belongs to the lower class $(\epsilon \mathcal{L})$ (with respect to $\{X_k\}$) if, with probability one, there exist infinitely many n such that

$$S_n > s_n \phi_n \, .$$

A sequence $\{\phi_n\}$ belongs to the upper class $(\epsilon \mathcal{U})$ if, with probability one, there exist only finitely many n such that

$$S_n > s_n \phi_n \, .$$

Using this terminology, the law of the iterated logarithm (Theorem 3, Ch. II) states :

If each X_k is bounded, and

$$\text{l.u.b.}\,|X_k| = o\left(\frac{s_n}{\sqrt{\log\log s_n}}\right) \tag{8}$$

uniformly in $k = 1, 2, \ldots, n$, then

$$\{(a \log\log s_n)^{\frac{1}{2}}\} \begin{array}{ll} \varepsilon\, \mathcal{U} & \text{if } a > 2 \\ \varepsilon\, \mathcal{L} & \text{if } a < 2. \end{array}$$

The following theorems give necessary and sufficient conditions for a sequence $\{\phi_n\}$ to belong to the upper or the lower class. In addition to assumptions (1)-(7), we also suppose that

$$2 < \phi_1 \leq \phi_2 \leq \phi_3 \leq \cdots . \tag{9}$$

We begin by replacing (8) with the stronger condition

$$|X_k| = O\left(\frac{s_n}{\phi_n^3}\right) .$$

<u>Theorem 1</u>

If

$$\lambda_n = O\left(\frac{1}{\phi_n^3}\right) , \tag{10}$$

then $\{\phi_n\}$ ε $\mathcal{U}(\mathcal{L})$ if and only if the series

$$\sum_n \frac{\sigma_n^2}{s_n^2} \phi_n \exp\left(- \frac{1}{2} \phi_n^2\right) \tag{11}$$

converges (diverges).

We now slightly relax the bound (10) on the X_k's.

<u>Theorem 2</u>

If

$$\lambda_n = O\left(\frac{1}{\phi_n^2}\right) \tag{12}$$

then $\{\phi_n\} \in \mathcal{U}(\mathcal{L})$ if and only if the series

$$\sum_n \frac{\sigma_n^2}{s_n^2} \phi_n \exp\left(- \frac{1}{2} \phi_n^2 - \frac{\phi_n^3}{6s_n^3} \sum_{k=1}^{n} b_k\right) \tag{13}$$

converges (diverges), where $b_k = \int_{-\infty}^{+\infty} x^3 dV_k(x)$.

Of course, if all of the X_k's are symmetric, then the third moments are all zero, giving the following immediate corollary.

<u>Corollary 1</u>

If $1 - V_k(x) = V_k(-x)$ and $\lambda_n = O\left(\frac{1}{\phi_n^2}\right)$, then $\{\phi_n\}$ is in the upper (lower) class if and only if the series (11) converges (diverges).

Notice that, in general, relaxing the bound on the X_k's from (10) to (12) leads to a criterion involving the third moments of the distribution functions $V_k(x)$. This is the beginning of a general pattern that is easily seen in the following theorem.

Theorem 3

If, for some integer $p \geq 1$,

$$\lambda_n = O\left(\frac{1}{\phi_n^{\frac{p+2}{p}}}\right), \tag{14}$$

then $\{\phi_n\} \varepsilon\, \mathcal{U}(\mathcal{L})$ if and only if the series

$$\sum_n \frac{\sigma_n^2}{s_n^2}\, \phi_n \exp[Q(\phi_n)]$$

converges (diverges), where $Q(x)$ is a polynomial of degree $p + 1$ whose jth coefficient depends only on the first j moments of $\{V_k(x)\}$.

Passing to the limit, we obtain the most general statement in the following form:

Theorem 4

If

$$\lambda_n \leq \frac{1}{200\ \phi_n} \tag{15}$$

then $\{\phi_n\} \varepsilon\, \mathcal{U}(\mathcal{L})$ if and only if the series

$$\sum_n \frac{\sigma_n^2}{s_n^2} \phi_n \exp[B_n(\phi_n)]$$

converges (diverges), where $B_n(x)$ is a power series whose jth coefficient depends only on the first j moments of $\{V_k(x)\}$.

The connection between these theorems and the law of the iterated logarithm becomes more apparent in the following theorem.

Theorem 5

If either

$$\lambda_n = o\left(\frac{1}{(\log\log s_n)^{\frac{3}{2}}}\right) ,$$

or $V_k(-x) = 1 - V_k(x)$ and

$$\lambda_n = o\left(\frac{1}{\log\log s_n}\right) ,$$

then

$$\phi_n = \{2\log_2 s_n^2 + 3\log_3 s_n^2 + 2\log_4 s_n^2 + \ldots$$

$$+ 2\log_{p-1} s_n^2 + (2+\delta)\log_p s_n^2\}^{\frac{1}{2}} \in \mathcal{U}(\mathcal{L})$$

if and only if $\delta > 0$ (< 0), where $\log_j x = \underbrace{\log\log\cdots\log}_{\text{j terms}} x$.

In many cases, the previous results can even be extended to sequences of unbounded random variables, as shown by Feller [2].

Many years after the work of Kolmogorov and Feller on the law of the iterated logarithm, some interesting new aspects of the law appeared in Strassen [8]. A brief indication of some of Strassen's results will now be given.

Suppose that $\{X_k\}$ is a sequence of independent identically distributed random variables with expected value zero and variance one. As usual, let $\{S_n\}$ be the corresponding sequence of partial sums. For each $n \geq 3$, let $\eta_n(t)$ be the continuous (random) function defined on the unit interval $[0,1]$ as follows :

$$\eta_n(0) = 0$$

$$\eta_n\left(\frac{i}{n}\right) = \frac{S_i}{\sqrt{2n \log \log n}} \qquad (i = 1, \ldots, n)$$

$\eta_n(t)$ linear on the intervals $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, $(i = 1, \ldots, n)$.

<u>Theorem 6</u>

With probability one, the set of functions

$$\{\eta_n(t) : n = 3, 4, \ldots\}$$

is conditionally compact in the uniform topology. Moreover, the set of limit points of the sequence consists precisely of those functions f(t) on [01] such that f(0) = 0, f is absolutely continuous on [01], and

$$\int_0^1 f'(t)\,dt \leq 1.$$

The proof of this theorem is quite technical. It consists first of proving analogous assertions for the Brownian motion process and then applying results of "invariance principle" type due to A. V. Skorokhod. For the proof, reference should be made to Strassen [8].

As a consequence of his principal result, Strassen establishes several related propositions. An example of these propositions is the following.

Theorem 7

Suppose that $0 \leq c \leq 1$. Let ξ_i be defined for $i \geq 3$ by

$$\xi_i = \begin{cases} 1 & \text{if } S_i > c(2i \log \log i)^{\frac{1}{2}} \\ 0 & \text{otherwise.} \end{cases}$$

Then, with probability one,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=3}^{n} \xi_i = 1 - \exp[\, - 4(\frac{1}{c^2} - 1)].$$

As a special instance of this result, consider the case

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